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A Complex Variable Approach to Electrochemical Machining Problems

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SUMMARY
Electrochemical machining is the name given to the process of eroding metal by electrolysis. The anode in this process is a workpiece from which metal is eroded and the cathode is a shaped machine tool which is fed towards the workpiece (see Fig. 1). Erosion takes place when an electric potential is applied across the electrolyte filled gap between the tool and the workpiece. The electrolyte is pumped through the gap in order to remove the products of erosion.

In this paper an attempt is made, under certain simplifying assumptions, to determine the resulting shape of the workpiece and the gap-sizes between the tool and workpiece. Basically two problems are treated in this paper; one for a plane-faced tool with complete insulation on the tool sides (see Fig. 2) and the other for an uninsulated straight sided tool (see Fig. 5). An exact complex variable technique is used and only minimal computer usage is required for final evaluations from derived analytic formulae.

1. Introduction

Electrochemical machining is the name given to the process of eroding metal by electrolysis. The anode in this process is a workpiece from which metal is eroded and the cathode is a shaped machine tool which is fed towards the workpiece (see Fig. 1). Erosion takes place when an electric potential is applied across the electrolyte filled gap between the tool and the workpiece. The electrolyte is pumped through the gap in order to remove the products of erosion.

In this paper an attempt is made, under certain simplifying assumptions, to determine the resulting shape of the workpiece and the gap-sizes between the tool and workpiece. The mathematical model considered assumes that the problem is two-dimensional and also assumes the steady state situation, in which the boundary of the workpiece moves in the direction of the tool feed at a constant rate equal to the tool feed rate. In addition changes in electrolyte conductivity which result from heating effects of the machining current, evolution of hydrogen gas, variations in hydraulic pressure etc. are assumed to be negligible and the conductivity is taken to be constant. Basically two problems are treated in this paper; one for a plane-

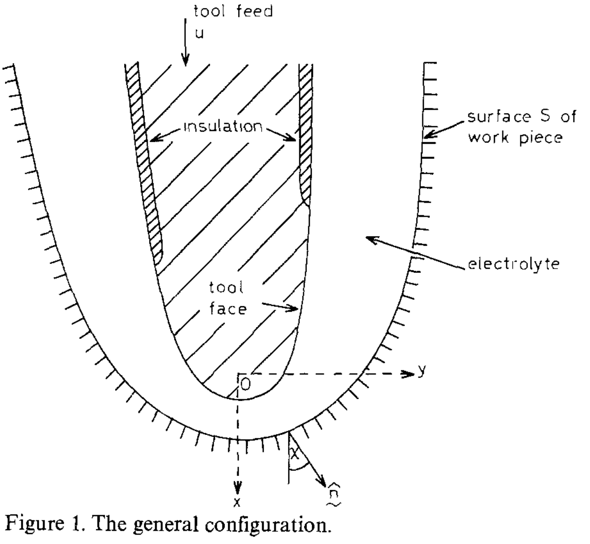

Figure 1. The general configuration.

faced tool with complete insulation on the tool sides (see Fig. 2) and the other for an uninsulated straight sided tool (see Fig. 5). An exact complex variable technique is used and only minimal computer usage is required for final evaluations from derived analytic formulae.

The problem as formulated in this paper has many features in common with problems connected with the magnetosphere, in which corpuscular flux of ions is incident on the earth's magnetic field [3]. Here the interest is to determine the shape of the magnetospheric boundary.

In [1], the authors consider the more general problem of a straight sided tool with side insulation extending to within a distance ω (the land width) of its leading edge. The method used is in two distinct parts; an approximate mathematical relation being first derived (their equation (58)) followed by a numerical solution of Laplace's equation in a region with an unknown boundary. It should be noted, however, that our results appear to differ significantly from those in [1].

Krylov [2] attempts to solve the problem in the "reverse" direction. Assuming a given workpiece shape the appropriate tool configuration is determined. As stated in [2] this approach in general does not lead to a well-posed mathematical problem and results are quoted for three rather "theoretical" shapes.

2. Formulation of the Problem

When the steady state has been reached, the problem may be reduced to a time-independent one by taking axes Oxy moving with the tool with the x-axis in the direction of the tool feed. Inside the electrolyte the electric field is

$$\boldsymbol{E} = (E_x, E_y, 0)\,. \tag{1}$$

Since div $\boldsymbol{E}=0$ and curl $\boldsymbol{E}=0$, we may use the methods of complex variable and take

$$\bar{E} \equiv E_x - iE_y = \frac{dw(z)}{dz}\,, \tag{2}$$

where $z=x+iy$ and $w=\phi+i\psi$. On the electrodes, the electric potential ϕ is a constant and we take

$$\phi = 0 \qquad \text{on the tool face,} \tag{3}$$

$$\phi = \phi_0\,(>0) \quad \text{on the unknown surface } S \text{ of the workpiece.} \tag{4}$$

On any insulated surfaces of the tool, the normal component of the electric field is zero and so

$$\psi = \text{constant on each of these surfaces}\,. \tag{5}$$

In the steady state, the boundary S moves in the direction of the tool feed at a constant rate equal to the tool feed rate u. The normal erosion rate, which is proportional to the electric current $\sigma(\partial\phi/\partial n)$, is therefore equal to $u \cos\chi$, where χ is the angle between the normal $\hat{n}$ to S and the x-axis. Thus, on S,

$$\frac{\partial\phi}{\partial n} = c \cos\chi\,, \tag{6}$$

where c is a constant. Now if s is the arc length measured along S in the direction of increasing y,

$$\cos\chi = \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial s}\,.$$

Upon integration, the boundary condition (6) may be reduced to

$$\psi = cy\,. \tag{7}$$

3. Solution for a Plane Faced Tool with Insulation

We now consider the case of a plane faced tool of width $2l$ with insulation extending behind the face as shown in Fig. 2. The configuration is symmetrical about the x-axis and since, from (7), $\psi = 0$ at A, we have

$$\psi = 0 \text{ on } OA . \tag{8}$$

It is sufficient to consider the region $OABCO$. As a consequence of (5), we suppose that

$$\psi = \psi_0 (>0) \text{ on } B_+C . \tag{9}$$

The boundary conditions in the w-plane are displayed in Fig. 3. In view of these conditions we

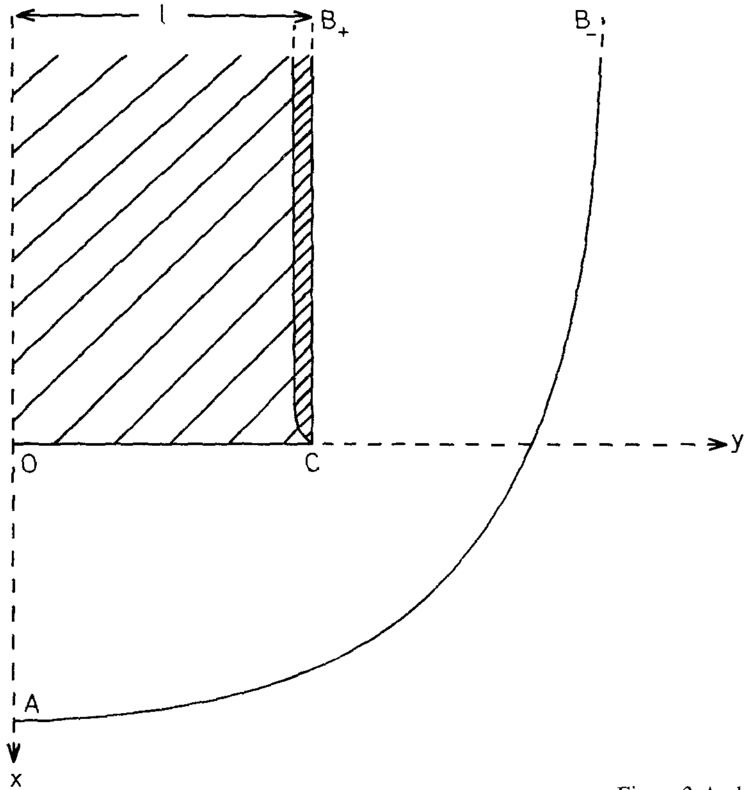

Figure 2. A plane faced tool with insulation.

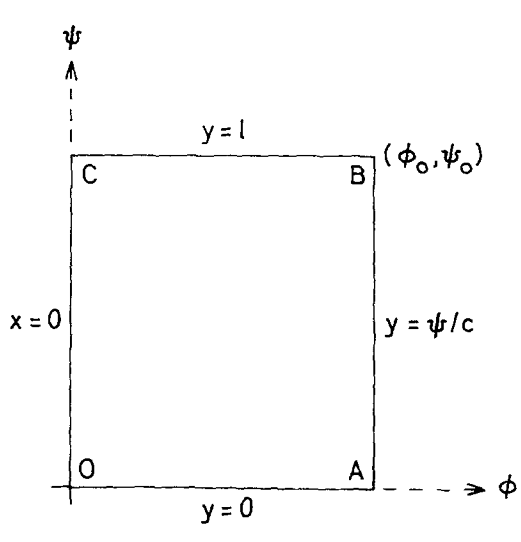

Figure 3. The boundary conditions in the w-plane.

look for a solution of the form

$$z = \frac{w}{c} + \sum_{n=0}^{\infty} a_n \sin\left[(2n+1)\frac{\pi w}{2\phi_0}\right], \tag{10}$$

where a_n are real constants. The conditions on OA, AB and CO are then satisfied. The condition on BC implies that

$$l = \frac{\psi_0}{c} + \sum_{n=0}^{\infty} a_n \cos\left[(2n+1)\frac{\pi\phi}{2\phi_0}\right] \sinh\left[(2n+1)\frac{\pi\psi_0}{2\phi_0}\right], \qquad 0 \leqslant \phi \leqslant \phi_0 .$$

Multiplying by $\cos\left[(2m+1)\frac{\pi\phi}{2\phi_0}\right]$ and integrating from $\phi=0$ to $\phi=\phi_0$ we obtain

$$a_n = \frac{8(l-\psi_0/c)(-1)^n q^{n+\frac{1}{2}}}{\pi(2n+1)(1-q^{2n+1})}, \tag{11}$$

where $q = \exp(-\pi\psi_0/\phi_0) < 1$. (12)

Thus

$$z = \frac{w}{c} + \frac{8}{\pi}\left(l - \frac{\psi_0}{c}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+\frac{1}{2}}}{(2n+1)(1-q^{2n+1})} \sin\left[(2n+1)\frac{\pi w}{2\phi_0}\right].$$

Now

$$\frac{dz}{dw} = \frac{1}{c} + \frac{4}{\phi_0}\left(l - \frac{\psi_0}{c}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+\frac{1}{2}}}{1-q^{2n+1}} \cos\left[(2n+1)\frac{\pi w}{2\phi_0}\right]$$

$$= \frac{1}{c} + \frac{2}{\phi_0}\left(l - \frac{\psi_0}{c}\right)\frac{Kk}{\pi} \operatorname{cd}\left(\frac{Kw}{\phi_0}\right), \tag{13}$$

where $\operatorname{cd}(Kw/\phi_0) \equiv \operatorname{cd}(Kw/\phi_0, k)$ is a Jacobi elliptic function with modulus k.

The mapping ceases to be conformal at C and we require

$$\frac{dz}{dw} = 0 \quad \text{when} \quad w = i\psi_0 . \tag{14}$$

Now

$$\frac{K\psi_0}{\phi_0} = \frac{K}{\pi}\log\left(\frac{1}{q}\right) = K'$$

and $\operatorname{cd}(iK') = k^{-1}$. Equations (13) and (14) then give

$$0 = \frac{1}{c} + \frac{2}{\phi_0}\left(l - \frac{\psi_0}{c}\right)\frac{K}{\pi} \tag{15}$$

and so

$$z = \frac{1}{c}\int_0^w \left[1 - k \operatorname{cd}\left(\frac{Kw}{\phi_0}\right)\right] dw$$

$$= \frac{w}{c} - \frac{\phi_0}{2cK}\log\left[\frac{1+k \operatorname{sn}(Kw/\phi_0)}{1-k \operatorname{sn}(Kw/\phi_0)}\right]. \tag{16}$$

When $w=\phi_0$, $z=h_e$ (say) where $h_e = OA$ is the equilibrium machine gap.

Now $\operatorname{sn}(K) = 1$ and so

$$h_e = \frac{\phi_0}{c}\left[1 - \frac{1}{2K}\log\left(\frac{1+k}{1-k}\right)\right]. \tag{17}$$

The total overcut $h_\infty = B_+ B_-$ is given by

$$h_\infty = \frac{\psi_0}{c} - l = \frac{\phi_0 \pi}{2cK} \tag{18}$$

where from (15)

$$\frac{\phi_0}{c} = \frac{lK}{K' - \frac{1}{2}\pi} . \tag{19}$$

The configuration is therefore completely specified by the parameters ϕ_0, c and l. The surface of the workpiece, given by (16) with $w = \phi_0 + icy$ is shown in Fig. 2, for the case $q = 0.002$. Here $\phi_0/c = 1.014l$, $h_e = 0.899l$ and $h_\infty = 1.006l$.

For practical purposes the machine gap is very small and for this case $q \ll 1$. In the limit as $q \to 0$, $k \to 0$, $K \to \pi/2$ and (17), (18) imply that

$$h_e = h_\infty = \phi_0/c . \tag{20}$$

The total overcut is then equal to the machine gap. In order to examine this limiting configuration near the side BC of the tool we put

$$z = il + \frac{2h_e}{\pi} z' , \tag{21}$$

$$w = i\psi_0 + \frac{2\phi_0}{\pi} w' , \tag{22}$$

where z' and w' are non-dimensional, and let $k \to 0$. Now as $k \to 0$

$$k \operatorname{sn}\left(\frac{Kw}{\phi_0}\right) \sim \frac{1}{\sin w'}$$

and (16) gives

$$z' = w' - \log\left(\frac{1 - i\, e^{iw'}}{1 + i\, e^{iw'}}\right) - \frac{i\pi}{2} . \tag{23}$$

The equation of the workpiece surface is now given by (23) with $w' = (\pi/2)(1-i) + iy'$ and is

$$x' = \frac{\pi}{2} - \log\left(\frac{e^{\pi/2 - y'} + 1}{e^{\pi/2 - y'} - 1}\right)$$

which may be written in the alternative form

$$e^{x' + y' - \pi} + e^{x' - \pi/2} + e^{y' - \pi/2} = 1 \tag{24}$$

showing that the work piece surface is symmetrical about the line $x' = y'$. This limiting configuration is shown in Fig. 4. It may be noted that when $x' = 0$

$$y' = \pi/2 - \log(\coth \pi/4)$$

and so

$$\frac{\text{overcut at corner}}{\text{machine gap}} = 1 - \frac{2}{\pi} \log\left(\coth \frac{\pi}{4}\right) \simeq 0.731 . \tag{25}$$

4. The Solution for the Case of a Straight Sided Tool with no Insulation

Here we consider a straight sided tool (symmetrical about the x-axis) with no insulation, as shown in Fig. 5, where we suppose that $\psi = \psi_0$ at C. Since, for practical purposes, the machine gap h_e is very small, we shall only consider the limiting case as $h_e \to 0$. In order to confine attention to the configuration near the side $B_+ C$ of the tool we again introduce the non-dimensional quantities z' and w' given by (21) and (22). As $h_e \to 0$, the side effects near OA are

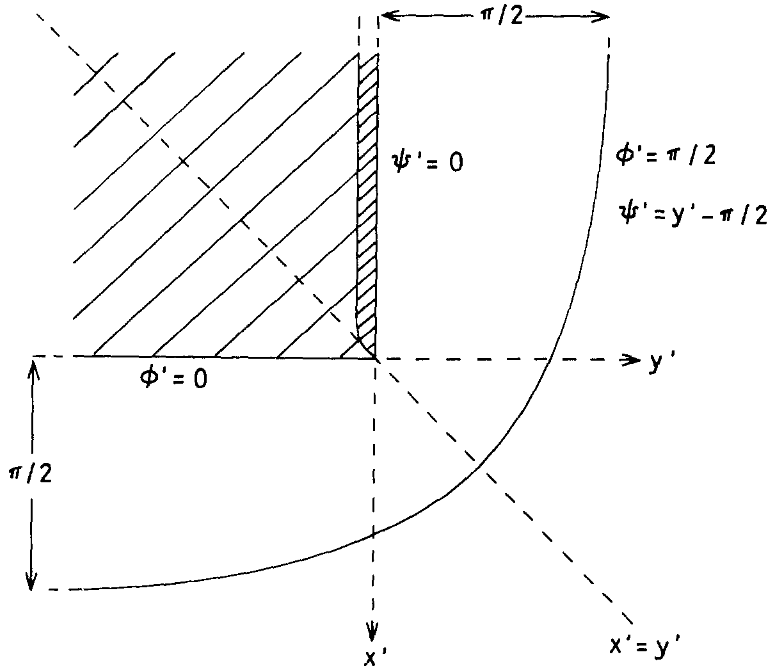

Figure 4. The limiting case for a plane faced tool with insulation.

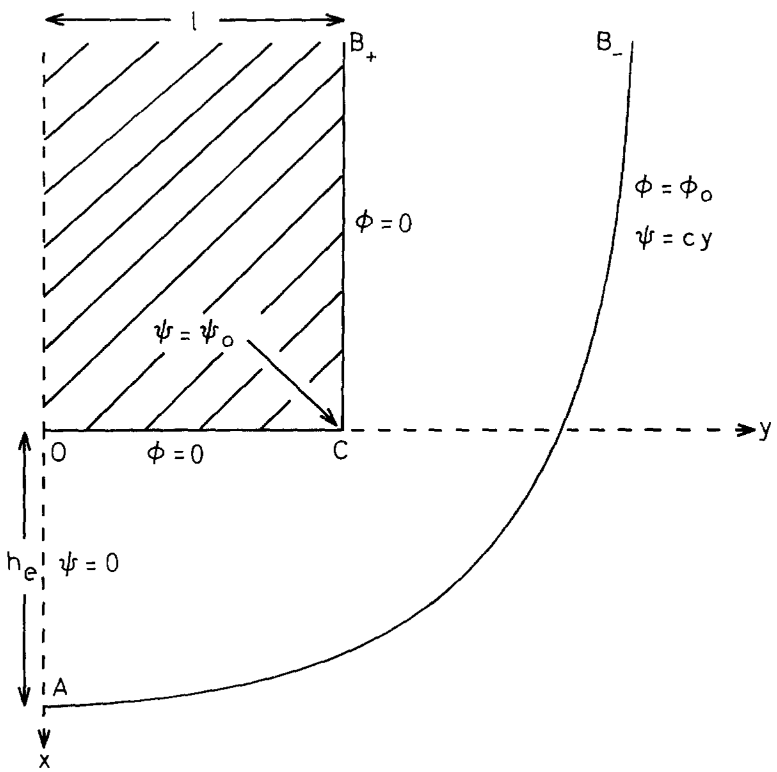

Figure 5. A sketch showing the configuration for a straight sided tool without insulation.

negligible and it is evident that $h_e \sim \phi_0/c$. The limiting configuration, together with the appropriate boundary conditions, is shown in Fig. 6 where

$$\alpha = \lim_{h_e \to 0} \left[\frac{\pi}{2\phi_0} (\psi_0 - cl) \right] \tag{26}$$

is to be determined.

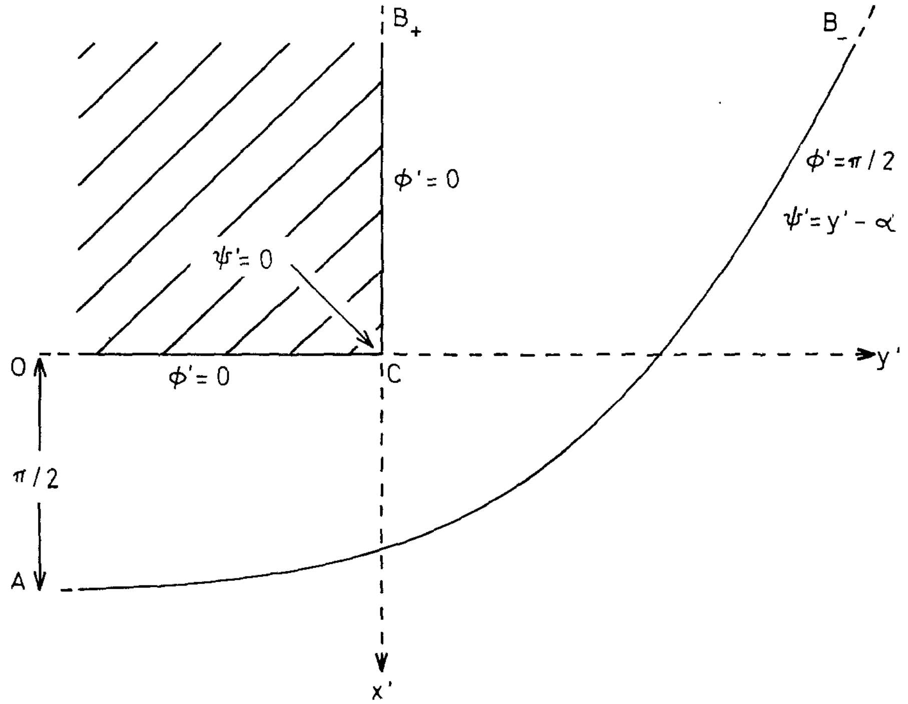

Figure 6. The limiting case for a straight sided tool without insulation.

Under the transformation

$$w' = -i \log(\cosh \zeta), \tag{27}$$

the w'-plane is mapped into the semi-infinite strip as shown in Fig. 7 where the boundary conditions are also displayed.

In view of these conditions we look for a solution of the form

$$z' = w' + i\alpha + i \sum_{n=0}^{\infty} b_n e^{-(2n+1)\zeta}, \tag{28}$$

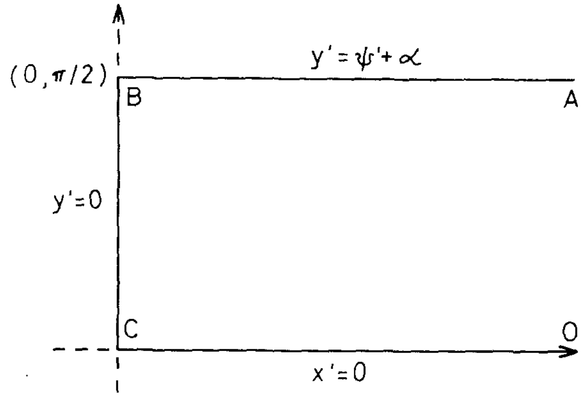

Figure 7. The boundary conditions in the ζ-plane.

where b_n are real constants. The conditions on AB and CO are then satisfied. On BC, $\zeta=i\eta$ (say), and so

$$\sum_{n=0}^{\infty} b_n \cos(2n+1)\eta = -\alpha-\psi' = -\alpha+\log(\cos\eta), \qquad 0\leqslant\eta\leqslant\frac{\pi}{2},$$

giving

$$b_n=\frac{4}{\pi}\left\{-\frac{\alpha(-1)^n}{2n+1}+\int_0^{\frac{1}{2}\pi}\log(\cos\eta)\cos(2n+1)\eta\,d\eta\right\}. \tag{29}$$

Substituting for b_n, (28) gives

$$z'=w'+i\alpha-\frac{2\alpha}{\pi}\log\left(\frac{1+ie^{-\zeta}}{1-ie^{-\zeta}}\right)+\frac{2i}{\pi}\sinh\zeta\int_0^{\frac{1}{2}\pi}\frac{\log(\cos\eta)\cos\eta\,d\eta}{\sinh^2\zeta+\sin^2\eta}$$

$$=w'+i\alpha-\frac{1}{\pi}(\alpha-\log 2)\log\left(\frac{\sinh\zeta+i}{\sinh\zeta-i}\right)+\frac{1}{\pi}\int_\infty^{\zeta}\tanh\zeta\log\left(\frac{\sinh\zeta+i}{\sinh\zeta-i}\right)d\zeta. \tag{30}$$

Now the mapping ceases to be conformal at C and so we require

$$\frac{dz'}{d\zeta}=0 \quad\text{when}\quad \zeta=0. \tag{31}$$

Using (30), this implies that

$$\alpha=\log 2, \tag{32}$$

and so

$$z'=w'+i\log 2+\frac{1}{\pi}\int_\infty^{\zeta}\tanh\zeta\log\left(\frac{\sinh\zeta+i}{\sinh\zeta-i}\right)d\zeta$$

$$=w'+i\log 2-\frac{2}{\pi}\int_0^{ie^{-iw'}}\frac{1}{p}\log\left[p+(p^2+1)^{\frac{1}{2}}\right]dp \tag{33}$$

using (27).

On the workpiece surface $w'=\pi/2+iy'-i\log 2$ and using (33), we obtain the equation for this surface, namely

$$x'=\frac{\pi}{2}-\frac{2}{\pi}\int_0^{\frac{1}{2}e^{y'}}\frac{1}{p}\log\left[p+(p^2+1)^{\frac{1}{2}}\right]dp. \tag{34}$$

As $x'\to-\infty$, $y'\to\infty$ and, as is to be expected, the total overcut is infinite. When $x'=0$, $y'=y^*$, where

$$\int_0^{\frac{1}{2}e^{y^*}}\frac{1}{p}\log\left[p+(p^2+1)^{\frac{1}{2}}\right]dp=\frac{\pi^2}{4}$$

giving

$$\frac{\text{overcut at corner}}{\text{machine gap}}=\frac{2y^*}{\pi}\simeq 1.159\,. \tag{35}$$

5. Conclusion

In a previous investigation [1], for a straight sided tool with land width ω, approximate methods produced the ratio

$$\frac{\text{overcut at corner}}{\text{machine gap}}\simeq 1.7\,.$$

The non-zero land width used in the paper is not specified but it seems probable that the authors have assumed this ratio to be insensitive to ω provided that ω is large compared with the machine gap. However, in the exact theory presented in this paper for infinite land width, the value is 1.159 and it seems reasonable to suppose that for a finite land width this ratio is smaller. It is hoped that these methods may be extended to include other, more general, configurations and in particular the case of finite non-zero land widths.

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